Quantum correlations IV

Wladyslaw Adam Majewski

Instytut Fizyki Teoretycznej i Astrofizyki, UG
ul. Wita Stwosza 57, 80-952 Gdańsk, Poland;

- *MATHEMATICAL TOOLS.*

- In this lecture we provide a brief exposition of the theory of tensor products and decomposition theory.

- Tensor products are necessary to define quantum composite systems, to define entangled density matrices.

- We remind that these concepts are indispensable for a definition of correlated quantities!

- Decomposable theory is necessary to decompose a state into more simple components.

- We begin with the definition of tensor product of linear spaces:

- **Definition 1.** *Let* X *and* Y *are linear spaces. We say that a form* $\varphi : X \times Y \to \mathbb{C}$ *is bilinear if*

$$\varphi(\lambda_1 x_1 + \lambda_2 x_2, \lambda_3 y_1 + \lambda_4 y_2) =$$

$$= \lambda_1 \lambda_3 \varphi(x_1, y_1) + \lambda_1 \lambda_4 \varphi(x_1, y_2) + \lambda_2 \lambda_3 \varphi(x_2, y_1) + \lambda_2 \lambda_4 \varphi(x_2, y_2)$$

where $x_1, x_2 \in X$*,* $y_1, y_2 \in Y$ *and* $\lambda_i \in \mathbb{C}$*, here* $i = 1, ..., 4$*.*

Let us denote the linear space of all such bilinear forms by $B(X, Y)$*.*

A simple tensor is a linear form $x \otimes y$ *on the linear space* $B(X, Y)$*, i.e.* $x \otimes y \in B(X, Y)'$*, such that:*

$$(x \otimes y)(A) = A(x, y) \tag{1}$$

for all $A \in B(X, Y)$*.*

The algebraic tensor product of two linear spaces X and Y, $X \odot Y \subset B(X, Y)'$, is the set of all linear, finite, combinations of simple tensors.

A typical $v \in X \odot Y$ is of the form:

$$v = \sum_{i=1}^{n} \lambda_i x_i \otimes y_i \tag{2}$$

where we emphasize that the decomposition given by (2) is not unique.

- The above construction can be applied to Banach spaces X and Y.
- BUT, the algebraic tensor product of X, Y is not automatically a Banach space.
- To obtain tensor product of Banach spaces which itself is a Banach space one must define a norm on $X \odot Y$.
- On $X \odot Y$ one can define various norms.
- It is natural to restrict oneself to norms satisfying: $\|x \otimes y\| = \|x\| \, \|y\|$. Such norms are said to be cross-norms.
- There are exceptional cases, where the cross-norm is uniquely defined.
- The most important case for Physics is that one when Banach spaces X and Y are Hilbert spaces and we want $X \otimes Y$ to be also a Hilbert space.

- However, in general, there are plenty of cross-norms on $X \odot Y$. To illustrate this phenomenon we give:

- **Example 2.** *Operator norm.*
Let $\mathcal{H}_i$ be a Hilbert space and $B(\mathcal{H}_i)$ denote the space of all bounded linear operators on $\mathcal{H}_i$ (for $i = 1, 2$). Then the operator norm on $B(\mathcal{H}_1) \odot B(\mathcal{H}_2) \subseteq B(\mathcal{H}_1 \otimes \mathcal{H}_2)$ is taken from that on $B(\mathcal{H}_1 \otimes \mathcal{H}_2)$. It has the cross-norm property. The closure of $B(\mathcal{H}_1) \odot B(\mathcal{H}_2)$ with respect to this operator norm will be denoted by $B(\mathcal{H}_1) \otimes B(\mathcal{H}_2)$ and called the tensor product of $B(\mathcal{H}_1)$ and $B(\mathcal{H}_2)$.

- We now define another important cross-norm on $X \odot Y$:

Example 3. *Projective norm.*
Let X and Y be Banach spaces. The projective norm π on $X \odot Y$ is defined by

$$\pi(v) = \inf \left\{ \sum_{i=1}^{n} \|x_i\| \, \|y_i\| : \; v = \sum_{i=1}^{n} x_i \otimes y_i \right\} \tag{3}$$

The completion of $X \odot Y$ with respect to the norm π is called the projective tensor product and is denoted as $X \otimes_\pi Y$.

- The importance of projective norm follows from the old Grothendieck result.

- **Theorem 4.** *Let* X *and* Y *be Banach spaces. Then, there exists an isometric isomorphism between the Banach space* $\mathfrak{B}(X,Y)$ *of all bounded bilinear functionals on* $X \times Y$ *and the space* $(X \otimes_\pi Y)^*$ *of all continuous linear functionals on* $(X \otimes_\pi Y)$ *given by*

$$\hat{\varphi}(x \otimes y) = \varphi(x,y) \tag{4}$$

where $\varphi \in \mathfrak{B}(X,Y)$, $x \in X$, *and* $y \in Y$.

- Moreover, the similar norm *the operator space projective norm* will be crucial in definition of entangled density matrices.

- This will follow from the following result:

- **Theorem 5.** *Let* $\mathfrak{M} \subseteq B(\mathcal{H})$ *and* $\mathfrak{N} \subseteq B(\mathcal{K})$ *be two von Neumann algebras. Denote by* $\mathfrak{M}_*$ *the predual of* $\mathfrak{M}$*, i.e. such Banach space that* $(\mathfrak{M}_*)^*$ *is isomorphic to* $\mathfrak{M}$*, i.e.* $(\mathfrak{M}_*)^* \cong \mathfrak{M}$*. There is an isometry*

$$(\mathfrak{M} \otimes \mathfrak{N})_* = \mathfrak{M}_* \otimes_\pi \mathfrak{N}_* \tag{5}$$

where the von Neumann algebra $\mathfrak{M} \otimes \mathfrak{N}$ *is the weak closure of the set* $\{A \otimes B; A \in \mathfrak{M}, B \in \mathfrak{N}\}$*. In particular,*

$$B(\mathcal{H} \otimes \mathcal{K})_* = B(\mathcal{H})_* \otimes_\pi B(\mathcal{K})_*. \tag{6}$$

- It is important to note that the closure in the above theorem is taken with respect to *the operator space projective norm!* and the operator space projective norm is, in general, different from the projective norm (see Effros, Ruan book).

- The projective tensor product gains in interest if we realize that Rules $1-3$ and 5 provide a nice example of application of this tensor product.

- To see his, let $\mathfrak{A}$ stand for algebra of observables.

- We assume $\mathfrak{A}$ is either a C^*-algebra or (when speaking about normal states) a W^*-algebra. Obviously, always, it is a Banach space.

- The set of all states $\mathfrak{S}$ is a subset of $\mathfrak{A}^*$ (and $\mathfrak{A}^*$ is also a Banach space) while the collection of all density matrices gives a subset of $\mathfrak{A}_*$ ($\mathfrak{A}_*$ is the predual of $\mathfrak{A}$, so it is also a Banach space).

- The Born interpretation, cf Rule 3, implies

$$\mathfrak{A} \times \mathfrak{S} \ni \langle A, \varphi \rangle \to \varphi(A) \in \mathbb{C} \tag{7}$$

where $\varphi(A)$ is interpreted as the expectation value of A at the state $\varphi \in \mathfrak{S}$.

- Thus the Born's interpretation of Quantum Mechanics gives an element of $\mathfrak{B}(\mathfrak{A}, \mathfrak{A}^*)$ since the form $\hat{A}(\cdot, \cdot)$ on $\mathfrak{A} \times \mathfrak{S}$ defined by (7) can be extended to the bilinear continuous form on $\mathfrak{A} \times \mathfrak{A}^*$ (or on $\mathfrak{A} \times \mathfrak{A}_*$, if one was interested in density matrices only).

- Clearly, (7) provides only one specific form on $\mathfrak{A} \times \mathfrak{A}^*$ (or on $\mathfrak{A} \times \mathfrak{A}_*$ respectively).

- However, it is crucial to note that Rule 3 combined with Rule 5 leads to the following recipe:

$$\mathfrak{A} \times \mathfrak{S} \ni \langle A, \varphi \rangle \rightarrow \varphi(T_t(A)) \in \mathbb{C} \tag{8}$$

where $T_t \in \{T_t\}$ is a dynamical map. Obviously, in this way we are getting the large collection of bilinear, continuous forms.

- On the other hand, Theorem 4 says

$$\mathfrak{B}(\mathfrak{A}, \mathfrak{A}^*) \cong (\mathfrak{A} \otimes_\pi \mathfrak{A}^*)^* \tag{9}$$

- If the set of states $\mathfrak{S}$ consists of normal states only (so, for example, the collection of density matrices in Dirac's formalism of quantum mechanics is relevant) then one can rewrite (9) as

$$\mathfrak{B}(\mathfrak{A}, \mathfrak{A}_*) \cong (\mathfrak{A} \otimes_\pi \mathfrak{A}_*)^* \tag{10}$$

- Next, we use the another identification (again due to Grothendieck results)
$$L(\mathfrak{A}, \mathfrak{A}) \cong (\mathfrak{A} \otimes_{\pi} \mathfrak{A}_*)^*, \tag{11}$$
where $L(\mathfrak{A}, \mathfrak{A})$ stands for the set of all bounded linear maps from $\mathfrak{A}$ to $\mathfrak{A}$,

- It is not difficult to see that
$$\mathfrak{B}(\mathfrak{A}, \mathfrak{A}_*) \cong L(\mathfrak{A}, \mathfrak{A}) \tag{12}$$

- Therefore, (8) gives not a large collection of continuous forms - **It gives the whole set of bilinear continuous forms** on $\mathfrak{A} \times \mathfrak{A}_*$, which are also positive, i.e. $\varphi(T_t(A)) \geq 0$ if $A \geq 0$ and $\varphi \geq 0$.

- In other words, *Grothendieck theory of tensor products is perfectly compatible with the quantization rules of quantum mechanics.*

- Few words on the theory of tensor product of C*-algebras.

- Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be C*-algebras with unit. Obviously, $\mathfrak{A}_1 \odot \mathfrak{A}_2$ can be constructed as before since $\mathfrak{A}_i$, $i = 1, 2$, is also a Banach space).

- As we wish to get a tensor product which is still a C*-algebra, we must define a C*-norm α on $\mathfrak{A}_1 \odot \mathfrak{A}_2$ i.e. a norm such that $\alpha(x^* x) = (\alpha(x))^2$ and $\alpha(xy) \leq \alpha(x)\alpha(y)$.

- Again, in general, there are plenty of such norms.

- As usually we will consider concrete C^*-algebras. Thus, we will use the operator norm. Consequently, the completion of the algebraic tensor product $\mathfrak{A}_1 \odot \mathfrak{A}_2$ with respect to the operator norm will be denoted by $\mathfrak{A}_1 \otimes \mathfrak{A}_2$.

- In some cases (for so called nuclear C*-algebras) the tensor product is uniquely defined.

- Nice examples of such algebras are provided by abelian algebras $\mathcal{A}$ (again, classical systems offer a great simplification) as well as $M_n(\mathbb{C})$, where $n < \infty$.

- We recall that toy models of quantum theory are based on models such that $M_n(\mathbb{C})$, where $n < \infty$.

- Decomposition theory.

- Contrary to the classical case, a state of a quantum system can be decomposed in many ways.

- The general idea of decomposition theory, applied to a convex compact subset K of states, $K \subseteq \mathfrak{S}$, is to express the complex structure of K as a sum of more simpler compounds.

- To this end, we wish to find a measure μ which is supported by a distinguished subset of states, for example by extremal points $Ext(K)$ of K. Then, we look for a decomposition of a state $\omega \in K$ in the form

$$\omega(A) = \int_K \omega'(A) d\mu(\omega') \tag{13}$$

  where A is an observable.

- We recall that for the classical case (we have seen this in the second lecture), Dirac's measures (so pure states) have played an important role in arguments leading to general form of two points correlation function.

- This explains why the case of a measure μ supported by the subset $Ext(K)$ is so important!

- The above formula (13) can be rewritten as

$$\hat{A}(\omega) = \int_K \hat{A}(\omega')d\mu(\omega'). \tag{14}$$

- This indicates that, in fact, we are studying the barycentric decompositions.

- The barycenter of a measure μ is defined as

Definition 6. *Let K be a compact convex subspace in locally compact space X and let μ be a positive non-zero measure on K. We say that*

$$b(\mu) = \mu\left(K\right)^{-1} \int_K x d\mu(x) \tag{15}$$

is a barycenter of a measure μ, where the integral is understood in the weak sense.

- We will need:

- **Proposition 7.** *Let $\mathfrak{A}$ be a C*-algebra (not necessary with unit) and let $B_{\mathfrak{A}}$ denote positive linear functionals on $\mathfrak{A}$ with norm less than or equal to one. Then $B_{\mathfrak{A}}$ is a convex, weakly $*$-compact subset of the dual $\mathfrak{A}^*$ whose extremal points are pure states.*

- and also:

  Proposition 8. *The set of states $\mathfrak{S}_{\mathfrak{A}}$ is convex but it is weakly $*$-compact if and only if $\mathfrak{A}$ has a unit. In the latter case the extreme points of $\mathfrak{S}_{\mathfrak{A}}$ are pure states. Thus, it follows from Krein-Milman theorem, that $\mathfrak{S}_{\mathfrak{A}} = \overline{conv}(\mathfrak{S}_{\mathfrak{A}}^P)$ where $\mathfrak{S}_{\mathfrak{A}}^p$ stands for the set of all pure states in $\mathfrak{S}_{\mathfrak{A}}$.*

- Let K be a compact set.

- **Definition 9.** *1. Let $M_+(K)$ denote the set of all positive (Radon) measures on K, (K is a compact set). The support of measure $\mu \in M_+(K)$ is defined as the smallest closed subset C of K such that $\mu(C) = \mu(K)$.*
  2. The measure μ is said to be pseudosupported by an arbitrary set $A \subseteq K$ if $\mu(B) = 0$ for all Baire sets B such that $B \cap A = \emptyset$

- Here, Baire sets can be considered as elements of such σ-algebra $\mathcal{F}$, which is the smallest σ-algebra such that all continuous functions are measurable.

- Warning! One can find a probability measure μ and a Borel subset A such that μ is pseudosupported by A, but $\mu(A) = 0$.

- Existence.

- Let K be a compact convex set, and $M_1(K)$ be the set of positive normalized, i.e probability, measures on K, which is a (weakly-$*$) compact.

- The barycenter $b(\mu)$ of a general measure μ exists.

- **Proposition 10.** *For a $\mu \in M_1(K)$ there exists a unique point b(μ) in the set K such that*

$$f(b(\mu)) = \int_K f(\omega')d\mu(\omega') \tag{16}$$

for all affine, continuous, real-valued functions f on K.

- There are nontrivial decompositions.

- **Proposition 11.** *Let K be a convex compact subset of a locally convex Hausdorff space. The following two conditions are equivalent:*

  1. *each $\omega \in K$ is the barycenter of a unique maximal measure,*
  2. *K is a simplex, what is equivalent that a system is a classical system! (what was discussed in the first lecture).*

- Thus, for quantum systems, a non pure states has nontrivial decompositions.

- Decompositions of states were considered in Quantum Field Theory and Quantum Statistical Physics in 70' of last century.

- To avoid measure theoretical problems one can assume:

- **Definition 12.** *Ruelle's SC condition*
  Let $\mathfrak{A}$ be a C-algebra with unit, and $\mathfrak{F}$ a subset of the state space $\mathfrak{S}_{\mathfrak{A}}$. $\mathfrak{F}$ is said to satisfy separability condition (SC) if there exists a sequence of sub-C*-algebras $\{\mathfrak{A}_n\}$ such that $\bigcup_{n=1}^{\infty} \mathfrak{A}_n$ is dense in $\mathfrak{A}$ and each $\mathfrak{A}_n$ contains a two-sided, closed, separable ideal $\mathcal{I}_n$ such that*

$$\mathfrak{F} = \{\omega,\ \omega \in \mathfrak{S}_{\mathfrak{A}},\ \|\omega|_{\mathcal{I}_n}\| = 1,\ n \geq 1\}.$$

- This condition is satisfied in most physical models, for example in all models discussed in the last lecture.

- The main result of the second part of the lecture.

- We need:

- **Definition 13.** *A face F of a compact convex set K is defined to be a convex subset of K with the following property: if $\omega \in F$ can be written as $\omega = \lambda_1\omega_1 + \lambda_2\omega_2$ where $\lambda_i \geq 0$, $i = 1, 2$, $\lambda_1 + \lambda_2 = 1$, ω_1 and ω_2 are in K then we have $\omega_i \in F$ $(i = 1, 2)$.*

- **Example 14.** *1. One point face, $F = \{\omega\} \subset K$ is an extremal point of K.*
  2. Let $\mathfrak{M}$ be a von Neumann algebra. Then $\mathfrak{S}^n_{\mathfrak{M}}$ is a face in $\mathfrak{S}_{\mathfrak{M}}$.

- The promised result is:

- **Theorem 15.** *Let $\mathfrak{A}$ be a C*-algebra with identity and ω be a state over $\mathfrak{A}$. There are measures μ (determined by structures induced by the state ω) over $\mathfrak{S}_{\mathfrak{A}}$ such that any μ is pseudosupported by pure states $Ext(\mathfrak{S}_{\mathfrak{A}})$. Moreover, if additionally ω is in a face F of $\mathfrak{S}_{\mathfrak{A}}$ satisfying the separability condition SC then the set of extremal points $Ext(F)$ of F, is a Baire subset of the pure states on $\mathfrak{A}$ and*

$$\mu(Ext(F)) = 1.$$

- To comment the theorem we make:

- **Remark 16.** *1. Obviously, the results stated in above Theorem are significant for non trivial faces, i.e. when F is consisting of more than one point.*

  2. Theorem 15 says that the strategy described at the beginning of this section is working, i.e. a state $\omega \in F$ can be decomposed into pure states.

  3. Measures appearing in Theorem 15 are of very special type. They are in the class of so called $orthogonal\ measures$.